For convenience we present here a brief review of some of the linear ODEs that will occur frequently in the sections and chapters that follow. The symbol α represents a constant.

Constant-coefficient equations	General solutions
$y' + \alpha y = 0$	$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0, \quad \alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0, \quad \alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, & \text{or} \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x \end{cases}$
Cauchy-Euler equation	**General solutions, $x > 0$**
$x^2 y'' + x y' - \alpha^2 y = 0, \quad \alpha \geq 0$	$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \\ y = c_1 + c_2 \ln x, & \alpha = 0 \end{cases}$
Parametric Bessel equation ($\nu = 0$)	**General solution, $x > 0$**
$x y'' + y' + \alpha^2 x y = 0,$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$
Legendre's equation ($n = 0, 1, 2, \ldots$)	**Particular solutions are polynomials**
$(1 - x^2) y'' - 2 x y' + n(n+1) y = 0,$	$y = P_0(x) = 1,$ $y = P_1(x) = x,$ $y = P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$

Consider the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0. \tag{2}$$

As we have already seen, there are three possible cases for the parameter λ: zero, negative, or positive; that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$. The solution of the DEs

$$y'' = 0, \quad \lambda = 0, \tag{3}$$

$$y'' - \alpha^2 y = 0, \quad \lambda = -\alpha^2, \tag{4}$$

$$y'' + \alpha^2 y = 0, \quad \lambda = \alpha^2, \tag{5}$$

are, in turn,

$$y = c_1 + c_2 x, \tag{6}$$

$$y = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \tag{7}$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x. \tag{8}$$

When the boundary conditions $X'(0) = 0$ and $X'(L) = 0$ are applied to each of these solutions, (6) yields $y = c_1$, (7) yields only $y = 0$, and (8) yields $y = c_1 \cos \alpha x$ *provided* that $\alpha = n\pi/L$, $n = 1, 2, 3, \ldots$. Since $y = c_1$ satisfies the DE in (3) and the boundary conditions for any *nonzero* choice of c_1, we conclude that $\lambda = 0$ is an eigenvalue. Thus the eigenvalues and corresponding eigenfunctions of the problem are $\lambda_0 = 0$, $y_0 = c_1$, $c_1 \neq 0$, and $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$, $n = 1, 2, \ldots$, $y_n = c_1 \cos(n\pi x/L)$, $c_1 \neq 0$. We can, if desired, take $c_1 = 1$ in each case. Note also that the eigenfunction $y_0 = 1$ corresponding to the eigenvalue $\lambda_0 = 0$ can be incorporated in the family $y_n = \cos(n\pi x/L)$ by permitting $n = 0$. The set $\{\cos(n\pi x/L)\}$, $n = 0, 1, 2, 3, \ldots$, is orthogonal on the interval $[0, L]$.

The problems in (1) and (2) are special cases of an important general two-point boundary value problem. Let p, q, r, and r' be real-valued functions continuous on an interval $[a, b]$, and let $r(x) > 0$ and $p(x) > 0$ for every x in the interval. Then

$$\textit{Solve:}\qquad \frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \tag{9}$$

$$\textit{Subject to:}\qquad A_1 y(a) + B_1 y'(a) = 0 \tag{10}$$

$$A_2 y(b) + B_2 y'(b) = 0 \tag{11}$$

is said to be a **regular Sturm-Liouville problem.** The coefficients in the boundary conditions (10) and (11) are assumed to be real and independent of λ. In addition, A_1 and B_1 are not both zero, and A_2 and B_2 are not both zero. The boundary-value problems in (1) and (2) are regular Sturm-Liouville problems. From (1) we can identify $r(x) = 1$, $q(x) = 0$, and $p(x) = 1$ in the differential equation (9); in boundary condition (10) we identify $a = 0$, $A_1 = 1$, $B_1 = 0$, and in (11), $b = L$, $A_2 = 1$, $B_2 = 0$. From (2) the identifications would be $a = 0$, $A_1 = 0$, $B_1 = 1$ in (10), $b = L$, $A_2 = 0$, $B_2 = 1$ in (11).

The differential equation (9) is linear and homogeneous. The boundary conditions in (10) and (11), both a linear combination of y and y' *equal to zero at a point*, are also **homogeneous.** A boundary condition such as $A_2 y(b) + B_2 y'(b) = C_2$, where C_2 is a nonzero constant, is **nonhomogeneous.** A boundary-value problem that consists of a homogeneous linear differential equation and homogeneous boundary conditions is, of course, said to be a homogeneous BVP; otherwise, it is nonhomogeneous. The boundary conditions (10) and (11) are referred to as **separated** since each condition involves only a single boundary point.

Because a regular Sturm-Liouville problem is a homogeneous BVP, it always possesses the trivial solution $y = 0$. However, this solution is of no interest to us. As in Example 1, in solving such a problem, we seek numbers λ (eigenvalues) and nontrivial solutions y that depend on λ (eigenfunctions).

Theorem 11.3 is a list of the more important of the many properties of the regular Sturm-Liouville problem. We shall prove only the last property.

THEOREM 11.3 **Properties of the Regular Sturm-Liouville Problem**

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 \cdots < \lambda_n < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

(b) For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent.

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

Let y_m and y_n be eigenfunctions corresponding to eigenvalues λ_m and λ_n, respectively. Then

$$\frac{d}{dx}[r(x)y'_m] + (q(x) + \lambda_m p(x))y_m = 0 \tag{12}$$

$$\frac{d}{dx}[r(x)y'_n] + (q(x) + \lambda_n p(x))y_n = 0. \tag{13}$$

Multiplying (12) by y_n and (13) by y_m and subtracting the two equations gives

$$(\lambda_m - \lambda_n)p(x)y_m y_n = y_m \frac{d}{dx}[r(x)y'_n] - y_n \frac{d}{dx}[r(x)y'_m].$$

Integrating this last result by parts from $x = a$ to $x = b$ then yields

$$(\lambda_m - \lambda_n)\int_a^b p(x)y_m y_n\,dx = r(b)[y_m(b)y'_n(b) - y_n(b)y'_m(b)] - r(a)[y_m(a)y'_n(a) - y_n(a)y'_m(a)]. \quad (14)$$

Now the eigenfunctions y_m and y_n must both satisfy the boundary conditions (10) and (11). In particular, from (10) we have

$$A_1 y_m(a) + B_1 y'_m(a) = 0$$
$$A_1 y_n(a) + B_1 y'_n(a) = 0.$$

For this system to be satisfied by A_1 and B_1, not both zero, the determinant of the coefficients must be zero:

$$y_m(a)y'_n(a) - y_n(a)y'_m(a) = 0.$$

A similar argument applied to (11) also gives

$$y_m(b)\,y'_n(b) - y_n(b)\,y'_m(b) = 0.$$

Since both members of the right-hand side of (14) are zero, we have established the orothogonality relation

$$\int_a^b p(x)y_m(x)y_n(x)\,dx = 0, \quad \lambda_m \neq \lambda_n. \quad (15)$$

Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0. \quad (16)$$

SOLUTION We proceed exactly as in the example by considering three cases in which the parameter λ could be zero, negative, or positive: $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$. The solutions of the DE for these values are listed in (3)–(5). For the cases $\lambda = 0$ and $\lambda = -\alpha^2 < 0$ we find that the BVP in (16) possesses only the trivial solution $y = 0$. For $\lambda = \alpha^2 > 0$ the general solution of the differential equation is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. Now the condition $y(0) = 0$ implies that $c_1 = 0$ in this solution, so we are left with $y = c_2 \sin \alpha x$. The second boundary condition $y(1) + y'(1) = 0$ is satisfied if

$$c_2 \sin \alpha + c_2 \alpha \cos \alpha = 0.$$

In view of the demand that $c_2 \neq 0$, the last equation can be written

$$\tan \alpha = -\alpha. \quad (17)$$

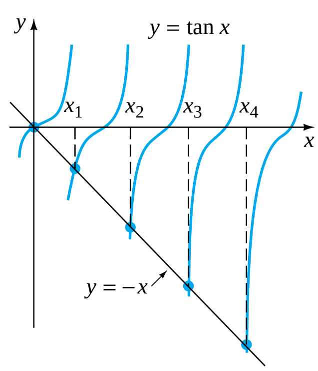

FIGURE 11.17 Positive roots $x_1, x_2, x_3, \ldots$ of $\tan x = -x$

If for a moment we think of (17) as $\tan x = -x$, then Figure 11.17 shows the plausibility that this equation has an infinite number of roots, namely, the x-coordinates of the points where the graph of $y = -x$ intersects the infinite number of branches of the graph of $y = \tan x$. The eigenvalues of the BVP (16) are then $\lambda_n = \alpha_n^2$, where α_n, $n = 1, 2, 3, \ldots$ are the consecutive *positive* roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ of (17). With the aid of a CAS it is easily shown that, to four rounded decimal places, $\alpha_1 = 2.0288$, $\alpha_2 = 4.9132$, $\alpha_3 = 7.9787$, and $\alpha_4 = 11.0855$, and the corresponding solutions are $y_1 = \sin 2.0288x$, $y_2 = \sin 4.9132x$, $y_3 = \sin 7.9787x$, and $y_4 = \sin 11.0855x$. In general, the eigenfunctions of the problem are $\{\sin \alpha_n x\}$, $n = 1, 2, 3, \ldots$.

With the identification $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, $A_1 = 1$, $B_1 = 0$, $A_2 = 1$, $B_2 = 1$ we see that (16) is a regular Sturm-Liouville problem. We conclude that $\{\sin \alpha_n x\}$, $n = 1, 2, 3, \ldots$, is an orthogonal set with respect to the weight function $p(x) = 1$ on the interval [0, 1].

In some circumstances we can prove the orthogonality of solutions of (9) without the necessity of specifying a boundary condition at $x = a$ and at $x = b$.

There are several other important conditions under which we seek nontrivial solutions of the differential equation (9):

- $r(a) = 0$, and a boundary condition of the type given in (11) is specified at $x = b$; (18)
- $r(b) = 0$, and a boundary condition of the type given in (10) is specified at $x = a$; (19)
- $r(a) = r(b) = 0$, and no boundary condition is specified at either $x = a$ or at $x = b$; (20)
- $r(a) = r(b)$, and boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$. (21)

The differential equation (9) along with one of conditions (18)–(20), is said to be a **singular** boundary-value problem. Equation (9) with the conditions specified in (21) is said to be a **periodic** boundary-value problem (the boundary conditions are also said to be periodic). Observe that if, say, $r(a) = 0$, then $x = a$ *may* be a singular point of the differential equation, and consequently, a solution of (9) may become unbounded as $x \to a$. However, we see from (14) that if $r(a) = 0$, then no boundary condition is required at $x = a$ to prove orthogonality of the eigenfunctions provided that these solutions are bounded at that point. This latter requirement guarantees the existence of the integrals involved. By assuming that the solutions of (9) are bounded on the closed interval $[a, b]$, we can see from inspection of (14) that

- if $r(a) = 0$, then the orthogonality relation (15) holds with no boundary condition specified at $x = a$; (22)
- if $r(b) = 0$, then the orthogonality relation (15) holds with no boundary condition specified at $x = b$;* (23)
- if $r(a) = r(b) = 0$, then the orthogonality relation (15) holds with no boundary conditions specified at either $x = a$ or $x = b$; (24)
- if $r(a) = r(b)$, then the orthogonality relation (15) holds with the periodic boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$. (25)

We note that a Sturm-Liouville problem is also singular when the interval under consideration is infinite. See Problems 9 and 10 in Exercises 11.4.

By carrying out the indicated differentiation in (9) we see that the differential equation is the same as

$$r(x)y'' + r'(x)y' + (q(x) + \lambda p(x))y = 0. \tag{26}$$

Examination of (26) might lead one to believe, given the coefficient of y' is the derivative of the coefficient of y'', that few differential equations have form (9). On the contrary, if the coefficients are continuous and $a(x) \neq 0$ for all x in some interval, then *any* second-order differential equation

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0 \tag{27}$$

can be recast into the so-called **self-adjoint form** (9). To this end we basically proceed as in Section 2.3, where we rewrote a homogeneous linear first-order equation $a_1(x)y' + a_0(x)y = 0$ in the form $\dfrac{d}{dx}[\mu y] = 0$ by dividing the equation by $a_1(x)$

and then multiplying by the integrating factor $\mu = e^{\int P(x)\,dx}$, where, assuming no common factors, $P(x) = a_0(x)/a_1(x)$. So first, we divide (27) by $a(x)$. The first two terms are $Y' + \dfrac{b(x)}{a(x)}Y + \cdots$, where for emphasis we have written $Y = y'$. Second, we multiply this equation by the integrating factor $e^{\int (b(x)/a(x))\,dx}$, where $a(x)$ and $b(x)$ are assumed to have no common factors:

$$\underbrace{e^{\int (b(x)/a(x))\,dx}Y' + \frac{b(x)}{a(x)}e^{\int (b(x)/a(x))\,dx}Y}_{\text{derivative of a product}} + \cdots = \frac{d}{dx}\left[e^{\int (b(x)/a(x))\,dx}Y\right] + \cdots = \frac{d}{dx}\left[e^{\int (b(x)/a(x))\,dx}y'\right] + \cdots .$$

In summary, by dividing (27) by $a(x)$ and then multiplying by $e^{\int (b(x)/a(x))\,dx}$, we get

$$e^{\int (b/a)\,dx}y'' + \frac{b(x)}{a(x)}e^{\int (b/a)\,dx}y' + \left(\frac{c(x)}{a(x)}e^{\int (b/a)\,dx} + \lambda\frac{d(x)}{a(x)}e^{\int (b/a)\,dx}\right)y = 0. \quad (28)$$

Equation (28) is the desired form given in (26) and is the same as (9):

$$\frac{d}{dx}\left[\underbrace{e^{\int (b/a)dx}}_{r(x)}y'\right] + \left(\underbrace{\frac{c(x)}{a(x)}e^{\int (b/a)dx}}_{q(x)} + \lambda\underbrace{\frac{d(x)}{a(x)}e^{\int (b/a)dx}}_{p(x)}\right)y = 0.$$

For example, to express $2y'' + 6y' + \lambda y = 0$ in self-adjoint form, we write $y'' + 3y' + \lambda\frac{1}{2}y = 0$ and then multiply by $e^{\int 3\,dx} = e^{3x}$. The resulting equation is

$$\overset{\overset{r(x)}{\downarrow}}{e^{3x}}y'' + \overset{\overset{r'(x)}{\downarrow}}{3e^{3x}}y' + \lambda\overset{\overset{p(x)}{\downarrow}}{\frac{1}{2}e^{3x}}y = 0 \quad \text{or} \quad \frac{d}{dx}\left[e^{3x}y'\right] + \lambda\frac{1}{2}e^{3x}y = 0.$$

It is certainly not necessary to put a second-order differential equation (27) into the self-adjoint form (9) to *solve* the DE. For our purposes we use the form given in (9) to determine the weight function $p(x)$ needed in the orthogonality relation (15).