

Suppose that f is a function defined on the interval $[-p, p]$ and can be expanded in an orthogonal series consisting of the trigonometric functions in the orthogonal set (1); that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right). \quad (2)$$

The coefficients $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ can be determined in exactly the same manner as in the general discussion of orthogonal series expansions on page 433. Before proceeding, note that we have chosen to write the coefficient of 1 in the set (1) as $\frac{1}{2} a_0$ rather than a_0 . This is for convenience only; the formula of a_n will then reduce to a_0 for $n = 0$.

Now integrating both sides of (2) from $-p$ to p gives

$$\int_{-p}^p f(x) dx = \frac{a_0}{2} \int_{-p}^p dx + \sum_{n=1}^{\infty} \left(a_n \int_{-p}^p \cos \frac{n\pi}{p} x dx + b_n \int_{-p}^p \sin \frac{n\pi}{p} x dx \right). \quad (3)$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \geq 1$ are orthogonal to 1 on the interval, the right side of (3) reduces to a single term:

$$\int_{-p}^p f(x) dx = \frac{a_0}{2} \int_{-p}^p dx = \frac{a_0}{2} x \Big|_{-p}^p = pa_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx. \quad (4)$$

Now we multiply (2) by $\cos(m\pi x/p)$ and integrate:

$$\begin{aligned} \int_{-p}^p f(x) \cos \frac{m\pi}{p} x dx &= \frac{a_0}{2} \int_{-p}^p \cos \frac{m\pi}{p} x dx \\ &+ \sum_{n=1}^{\infty} \left(a_n \int_{-p}^p \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x dx + b_n \int_{-p}^p \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x dx \right). \end{aligned} \quad (5)$$

By orthogonality we have

$$\int_{-p}^p \cos \frac{m\pi}{p} x dx = 0, \quad m > 0, \quad \int_{-p}^p \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x dx = 0,$$

$$\text{and} \quad \int_{-p}^p \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x dx = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases}$$

$$\text{Thus (5) reduces to} \quad \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx = a_n p,$$

$$\text{and so} \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx. \quad (6)$$

Finally, if we multiply (2) by $\sin(m\pi x/p)$, integrate, and make use of the results

$$\int_{-p}^p \sin \frac{m\pi}{p} x dx = 0, \quad m > 0, \quad \int_{-p}^p \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x dx = 0,$$

$$\text{and} \quad \int_{-p}^p \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x dx = \begin{cases} 0, & m \neq n \\ p, & m = n, \end{cases}$$

$$\text{we find that} \quad b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx. \quad (7)$$

The trigonometric series (2) with coefficients a_0 , a_n , and b_n defined by (4), (6), and (7), respectively, is said to be the **Fourier series** of the function f . The coefficients obtained from (4), (6), and (7) are referred to as **Fourier coefficients** of f .

In finding the coefficients a_0 , a_n , and b_n , we assumed that f was integrable on the interval and that (2), as well as the series obtained by multiplying (2) by $\cos(m\pi x/p)$, converged in such a manner as to permit term-by-term integration. Until (2) is shown to be convergent for a given function f , the equality sign is not to be taken in a strict or literal sense. Some texts use the symbol \sim in place of $=$. In view of the fact that most functions in applications are of a type that guarantees convergence of the series, we shall use the equality symbol. We summarize the results:

DEFINITION 11.5 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right), \quad (8)$$

where
$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx \quad (9)$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx \quad (10)$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx. \quad (11)$$

The following theorem gives sufficient conditions for convergence of a Fourier series at a point.

THEOREM 11.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $(-p, p)$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.*

Observe that each of the functions in the basic set (1) has a different fundamental period*—namely, $2p/n$, $n \geq 1$ —but since a positive integer multiple of a period is also a period, we see that all of the functions have in common the period $2p$. (Verify.) Hence the right-hand side of (2) is $2p$ -periodic; indeed, $2p$ is the **fundamental period** of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$, but also gives the **periodic extension** of f outside this interval. We can now apply Theorem 11.1 to the periodic extension of f , or we may assume from the outset that the given function is periodic with period $2p$; that is, $f(x + 2p) = f(x)$. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series (8) converges to the average

$$\frac{f(p-) + f(-p+)}{2}$$

at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, and so on.

The Fourier series in (13) converges to the periodic extension of (12) on the entire x -axis. At $0, \pm 2\pi, \pm 4\pi, \dots$ and at $\pm\pi, \pm 3\pi, \pm 5\pi, \dots$ the series converges to the values

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi-) + f(-\pi+)}{2} = 0,$$

respectively. The solid dots in Figure 11.2 represent the value $\pi/2$.

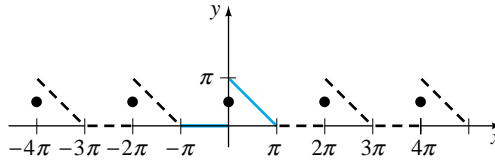


FIGURE 11.2 Periodic extension of function shown in Figure 11.1

It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example, the first three partial sums of (13) are

$$S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x, \quad \text{and} \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x + \frac{1}{2} \sin 2x.$$

In Figure 11.3 we have used a CAS to graph the partial sums $S_3(x)$, $S_8(x)$, and $S_{15}(x)$ of (13) on the interval $(-\pi, \pi)$. Figure 11.3(d) shows the periodic extension using $S_{15}(x)$ on $(-4\pi, 4\pi)$.

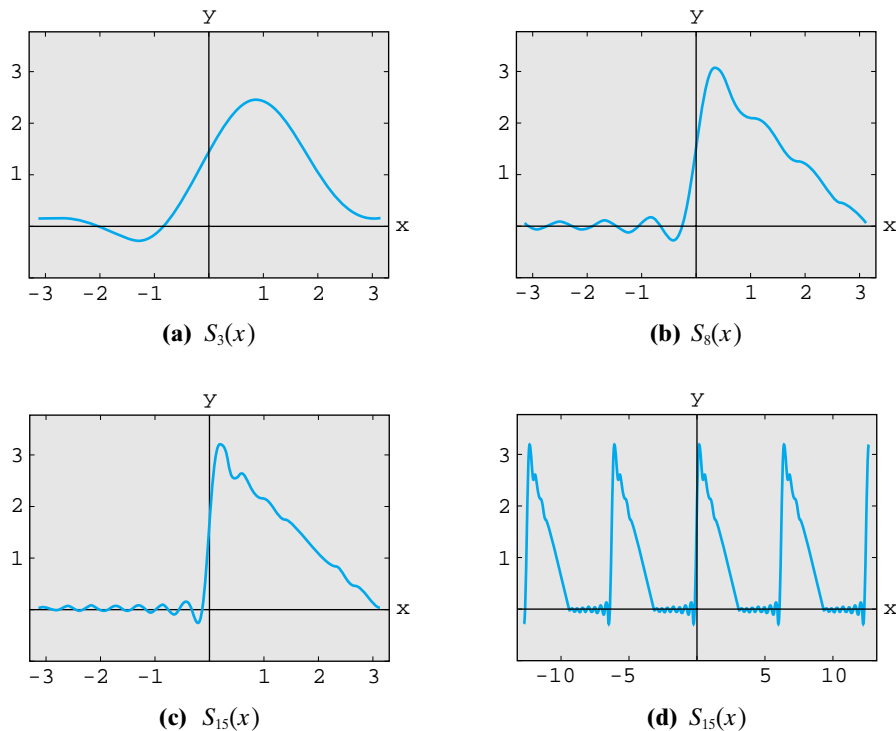


FIGURE 11.3 Partial sums of a Fourier series