Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\,\dfrac{\partial u}{\partial y}$.

SOLUTION Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields

$$X''Y = 4XY'.$$

After dividing both sides by $4XY$, we have separated the variables:

$$\frac{X''}{4X} = \frac{Y'}{Y}.$$

Since the left-hand side of the last equation is independent of y and is equal to the right-hand side, which is independent of x, we conclude that both sides of the equation are independent of x *and* y. In other words, each side of the equation must be a constant. In practice it is *convenient* to write this real **separation constant** as $-\lambda$ (using λ would lead to the same solutions).

From the two equalities

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

we obtain the two linear ordinary differential equations

$$X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0. \tag{2}$$

Now, we consider three cases for λ: zero, negative, or positive, that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$.

If $\lambda = 0$, then the two ODEs in (2) are

$$X'' = 0 \quad \text{and} \quad Y' = 0.$$

Solving each equation (by, say, integration), we find $X = c_1 + c_2x$ and $Y = c_3$. Thus a particular product solution of the given PDE is

$$u = XY = (c_1 + c_2x)c_3 = A_1 + B_1x, \tag{3}$$

where we have replaced c_1c_3 and c_2c_3 by A_1 and B_1, respectively.

If $\lambda = -\alpha^2$, then the DEs in (2) are

$$X'' - 4\alpha^2X = 0 \quad \text{and} \quad Y' - \alpha^2Y = 0.$$

From their general solutions

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \quad \text{and} \quad Y = c_6e^{\alpha^2 y}$$

we obtain another particular product solution of the PDE,

$$u = XY = (c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x)c_6e^{\alpha^2 y}$$

or

$$u = A_2e^{\alpha^2 y} \cosh 2\alpha x + B_2e^{\alpha^2 y} \sinh 2\alpha x, \tag{4}$$

where $A_2 = c_4c_6$ and $B_2 = c_5c_6$.

If $\lambda = \alpha^2$, then the DEs

$$X'' + 4\alpha^2X = 0 \quad \text{and} \quad Y' + \alpha^2Y = 0$$

and their general solutions

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \quad \text{and} \quad Y = c_9e^{-\alpha^2 y}$$

give yet another particular solution

$$u = A_3e^{-\alpha^2 y} \cos 2\alpha x + B_3e^{-\alpha^2 y} \sin 2\alpha x, \tag{5}$$

where $A_3 = c_7c_9$ and $B_2 = c_8c_9$.

The following theorem is analogous to Theorem 4.2 and is known as the **superposition principle.**

THEOREM 12.1 Superposition Principle

If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k,$$

where the c_i, $i = 1, 2, \ldots, k$, are constants, is also a solution.

Throughout the remainder of the chapter we shall assume that whenever we have an infinite set $u_1, u_2, u_3, \ldots$ of solutions of a homogeneous linear equation, we can construct yet another solution u by forming the infinite series

$$u = \sum_{k=1}^{\infty} c_k u_k,$$

where the c_i, $i = 1, 2, \ldots$ are constants.

A linear second-order partial differential equation in two independent variables with constant coefficients can be classified as one of three types. This classification depends only on the coefficients of the second-order derivatives. Of course, we assume that at least one of the coefficients A, B, and C is not zero.

DEFINITION 12.1 Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

where A, B, C, D, E, and F are real constants, is said to be

hyperbolic if $B^2 - 4AC > 0$,
parabolic if $B^2 - 4AC = 0$,
elliptic if $B^2 - 4AC < 0$.

Classify the following equations:

(a) $3\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$ **(b)** $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$ **(c)** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

SOLUTION **(a)** By rewriting the given equation as

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

we can make the identifications $A = 3$, $B = 0$, and $C = 0$. Since $B^2 - 4AC = 0$, the equation is parabolic.

(b) By rewriting the equation as

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

we see that $A = 1$, $B = 0$, $C = -1$, and $B^2 - 4AC = -4(1)(-1) > 0$. The equation is hyperbolic.

(c) With $A = 1$, $B = 0$, $C = 1$, and $B^2 - 4AC = -4(1)(1) < 0$, the equation is elliptic.