

FIGURE 12.5 Temperatures in a rod of length L

To start, we use the product $u(x, t) = X(x)T(t)$ to separate variables in (1). Then, if $-\lambda$ is the separation constant, the two equalities

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad (4)$$

lead to the two ordinary differential equations

$$X'' + \lambda X = 0 \quad (5)$$

$$T' + k\lambda T = 0. \quad (6)$$

Before solving (5), note that the boundary conditions (2) applied to $u(x, t) = X(x)T(t)$ are

$$u(0, t) = X(0)T(t) = 0 \quad \text{and} \quad u(L, t) = X(L)T(t) = 0.$$

Since it makes sense to expect that $T(t) \neq 0$ for all t , the foregoing equalities hold only if $X(0) = 0$ and $X(L) = 0$. These homogeneous boundary conditions together with the homogeneous DE (5) constitute a regular Sturm-Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \quad (7)$$

The solution of this BVP was thoroughly discussed in Example 2 of Section 5.2. In that example we considered three possible cases for the parameter λ : zero, negative, or positive. The corresponding solutions of the DEs are, in turn, given by

$$X(x) = c_1 + c_2 x, \quad \lambda = 0 \quad (8)$$

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \quad \lambda = -\alpha^2 < 0 \quad (9)$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \quad \lambda = \alpha^2 > 0. \quad (10)$$

When the boundary conditions $X(0) = 0$ and $X(L) = 0$ are applied to (8) and (9), these solutions yield only $X(x) = 0$, and so we would have to conclude that $u = 0$. But when $X(0) = 0$ is applied to (10), we find that $c_1 = 0$ and $X(x) = c_2 \sin \alpha x$. The second boundary condition then implies that $X(L) = c_2 \sin \alpha L = 0$. To obtain a nontrivial solution, we must have $c_2 \neq 0$ and $\sin \alpha L = 0$. The last equation is satisfied when $\alpha L = n\pi$ or $\alpha = n\pi/L$. Hence (7) possesses nontrivial solutions when $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$, $n = 1, 2, 3, \dots$. These values of λ are the **eigenvalues** of the problem; the **eigenfunctions** are

$$X(x) = c_2 \sin \frac{n\pi}{L} x, \quad n = 1, 2, 3, \dots \quad (11)$$

From (6) we have $T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$, so

$$u_n = X(x)T(t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x, \quad (12)$$

where we have replaced the constant $c_2 c_3$ by A_n . Each of the product functions $u_n(x, t)$ given in (12) is a particular solution of the partial differential equation (1), and each $u_n(x, t)$ satisfies both boundary conditions (2) as well. However, for (12) to satisfy the initial condition (3), we would have to choose the coefficient A_n in such a manner that

$$u_n(x, 0) = f(x) = A_n \sin \frac{n\pi}{L} x. \quad (13)$$

In general, we would not expect condition (13) to be satisfied for an arbitrary, but reasonable, choice of f . Therefore we are forced to admit that $u_n(x, t)$ is *not* a solution of the given problem. Now by the superposition principle (Theorem 12.1) the function $u(x, t) = \sum_{n=1}^{\infty} u_n$ or

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x \quad (14)$$

must also, although formally, satisfy equation (1) and the conditions in (2). Substituting $t = 0$ into (14) implies that

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x.$$

This last expression is recognized as a half-range expansion of f in a sine series. If we make the identification $A_n = b_n$, $n = 1, 2, 3, \dots$, it follows from (5) that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx. \quad (15)$$

We conclude that a solution of the boundary-value problem described in (1), (2), and (3) is given by the infinite series

$$u(x, t) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \sin \frac{n\pi}{L} x dx \right) e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x. \quad (16)$$

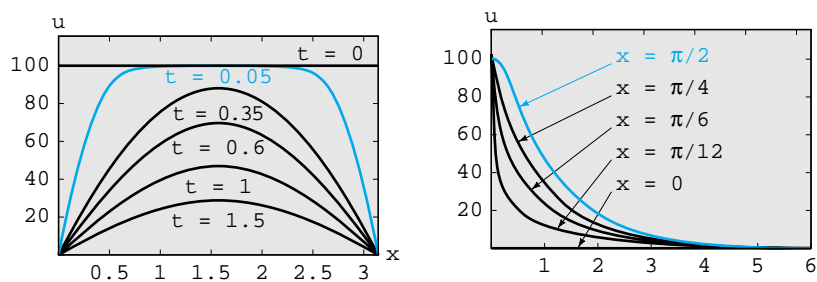
In the special case when the initial temperature is $u(x, 0) = 100$, $L = \pi$, and $k = 1$, you should verify that the coefficients (15) are given by

$$A_n = \frac{200}{\pi} \left[\frac{1 - (-1)^n}{n} \right],$$

and that (16) is

$$u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n} \right] e^{-n^2 t} \sin nx. \quad (17)$$

Since u is a function of two variables, the graph of the solution (17) is a surface in 3-space. We could use the 3D-plot application of a computer algebra system to approximate this surface by graphing partial sums $S_n(x, t)$ over a rectangular region defined by $0 \leq x \leq \pi$, $0 \leq t \leq T$. Alternatively, with the aid of the 2D-plot application of a CAS we can plot the solution $u(x, t)$ on the x -interval $[0, \pi]$ for increasing values of time t . See Figure 12.6(a). In Figure 12.6(b) the solution $u(x, t)$ is graphed on the t -interval $[0, 6]$ for increasing values of x ($x = 0$ is the left end and $x = \pi/2$ is the midpoint of the rod of length $L = \pi$.) Both sets of graphs verify what is apparent in (17)—namely, $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$.



(a) $u(x, t)$ graphed as a function of x for various fixed times

(b) $u(x, t)$ graphed as a function of t for various fixed positions

FIGURE 12.6 Graphs of (17) when one variable is held fixed