

FIGURE 12.13 Steady-state temperatures in a rectangular plate

With $u(x, y) = X(x)Y(y)$ separation of variables in (1) leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \quad (4)$$

$$Y'' - \lambda Y = 0. \quad (5)$$

The three homogeneous boundary conditions in (2) and (3) translate into $X'(0) = 0$, $X'(a) = 0$, and $Y(0) = 0$. The Sturm-Liouville problem associated with the equation in (4) is then

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(a) = 0. \quad (6)$$

Examination of the cases corresponding to $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$, has already been carried out in an example from before. Here is a brief summary of that analysis.

For $\lambda = 0$, (6) becomes

$$X'' = 0, \quad X'(0) = 0, \quad X'(a) = 0.$$

The solution of the DE is $X = c_1 + c_2x$. The boundary conditions imply $X = c_1$. By imposing $c_1 \neq 0$, this problem possesses a nontrivial solution. For $\lambda = -\alpha^2 < 0$, (6) possesses only the trivial solution. For $\lambda = \alpha^2 > 0$, (6) becomes

$$X'' + \alpha^2 X = 0, \quad X'(0) = 0, \quad X'(a) = 0.$$

The solution of the DE in this problem is $X = c_1 \cos \alpha x + c_2 \sin \alpha x$. The boundary condition $X'(0) = 0$ implies that $c_2 = 0$, so $X = c_1 \cos \alpha x$. Differentiating this last expression and then setting $x = a$ gives $-c_1 \sin \alpha a = 0$. Since we have assumed that $\alpha > 0$, this last condition is satisfied when $\alpha a = n\pi$ or $\alpha = n\pi/a$, $n = 1, 2, \dots$. The eigenvalues of (6) are then $\lambda_0 = 0$ and $\lambda_n = \alpha_n^2 = n^2\pi^2/a^2$, $n = 1, 2, \dots$. If we correspond $\lambda_0 = 0$ with $n = 0$, the eigenfunctions of (6) are

$$X = c_1, \quad n = 0, \quad \text{and} \quad X = c_1 \cos \frac{n\pi}{a}x, \quad n = 1, 2, \dots$$

We now solve equation (5) subject to the single homogeneous boundary condition $Y(0) = 0$. There are two cases. For $\lambda_0 = 0$, equation (5) is simply $Y'' = 0$; therefore its solution is $Y = c_3 + c_4y$. But $Y(0) = 0$ implies that $c_3 = 0$, so $Y = c_4y$. For $\lambda_n = n^2\pi^2/a^2$, (5) is $Y'' - \frac{n^2\pi^2}{a^2}Y = 0$. Because $0 < y < b$ is a finite interval, we use (according to the informal rule indicated on page 448) the hyperbolic form of the general solution:

$$Y = c_3 \cosh(n\pi y/a) + c_4 \sinh(n\pi y/a).$$

$Y(0) = 0$ again implies that $c_3 = 0$, so we are left with $Y = c_4 \sinh(n\pi y/a)$.

Thus product solutions $u_n = X(x)Y(y)$ that satisfy the Laplace's equation (1) and the three homogeneous boundary conditions in (2) and (3) are

$$A_0 y, \quad n = 0, \quad \text{and} \quad A_n \sinh \frac{n\pi}{a}y \cos \frac{n\pi}{a}x, \quad n = 1, 2, \dots,$$

where we have rewritten c_1c_4 as A_0 for $n = 0$ and as A_n for $n = 1, 2, \dots$

The superposition principle yields another solution:

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x. \quad (7)$$

We are now in a position to use the last boundary condition in (3). Substituting $x = b$ in (7) gives

$$u(x, b) = f(x) = A_0 b + \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi}{a} b \right) \cos \frac{n\pi}{a} x,$$

which is a half-range expansion of f in a cosine series. If we make the identifications $A_0 b = a_0/2$ and $A_n \sinh(n\pi b/a) = a_n$, $n = 1, 2, 3, \dots$, it follows from (2) and (3) of Section 11.3 that

$$2A_0 b = \frac{2}{a} \int_0^a f(x) dx$$

$$A_0 = \frac{1}{ab} \int_0^a f(x) dx \quad (8)$$

and

$$A_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi}{a} x dx$$

$$A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \cos \frac{n\pi}{a} x dx. \quad (9)$$

The solution of the boundary-value problem (1)–(3) consists of the series in (7), with coefficients A_0 and A_n defined in (8) and (9), respectively.

A boundary-value problem in which we seek a solution of an elliptic partial differential equation such as Laplace's equation $\nabla^2 u = 0$, within a bounded region R (in the plane or 3-space) such that u takes on prescribed values on the entire boundary of the region is called a **Dirichlet problem**. In another problem you are asked to show that the solution of the Dirichlet problem for a rectangular region

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b$$

$$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 < x < a$$

is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x, \quad \text{where} \quad A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \sin \frac{n\pi}{a} x dx. \quad (10)$$

In the special case when $f(x) = 100$, $a = 1$, $b = 1$, the coefficients A_n in (10) are given by $A_n = 200 \frac{1 - (-1)^n}{n\pi \sinh n\pi}$. With the help of a CAS we plotted the surface defined by $u(x, y)$ over the region R : $0 \leq x \leq 1$, $0 \leq y \leq 1$, in Figure 12.14(a). You can see in the figure that the boundary conditions are satisfied; especially note that along $y = 1$, $u = 100$ for $0 \leq x \leq 1$. The **isotherms**, or curves in the rectangular region along which the temperature $u(x, y)$ is constant, can be obtained by using the contour plotting capabilities of a CAS and are illustrated in Figure 12.14(b). The isotherms can also be visualized as the curves of intersection (projected into the xy -plane) of horizontal planes $u = 80$, $u = 60$, and so on, with the surface in Figure 12.14(a). Notice that throughout the region the maximum temperature is $u = 100$ and occurs on the portion of the boundary corresponding to $y = 1$. This is no coincidence. There is a **maximum principle** that states a solution u of Laplace's equation within a bounded region R with boundary B (such as a rectangle, circle, sphere, and so on) takes on its maximum and minimum values on B . In addition, it can be proved that u can have no relative extrema (maxima or

minima) in the interior of R . This last statement is clearly borne out by the surface shown in Figure 12.14(a).

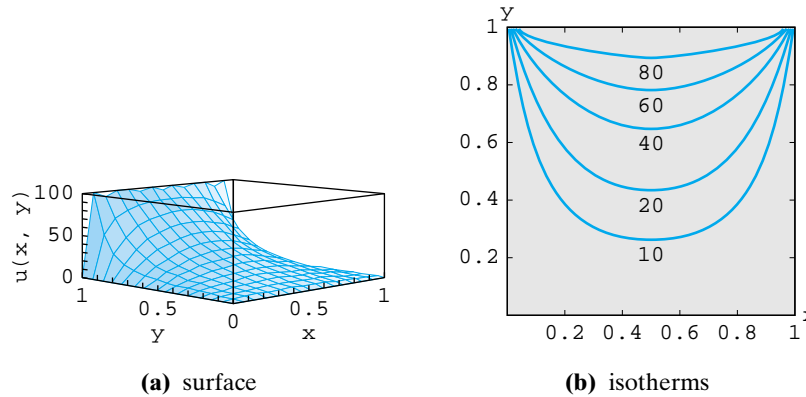


FIGURE 12.14 Surface is graph of partial sums when $f(x) = 100$ and $a = b = 1$ in (10)

A Dirichlet problem for a rectangle can be readily solved by separation of variables when homogeneous boundary conditions are specified on two *parallel* boundaries. However, the method of separation of variables is not applicable to a Dirichlet problem when the boundary conditions on all four sides of the rectangle are nonhomogeneous. To get around this difficulty, we break the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, & 0 < x < a, & \quad 0 < y < b \\ u(0, y) &= F(y), & u(a, y) &= G(y), & 0 < y < b \\ u(x, 0) &= f(x), & u(x, b) &= g(x), & 0 < x < a \end{aligned} \quad (11)$$

into two problems, each of which has homogeneous boundary conditions on parallel boundaries, as shown:

Problem 1	Problem 2
$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$	$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$
$u_1(0, y) = 0, \quad u_1(a, y) = 0, \quad 0 < y < b$	$u_2(0, y) = F(y), \quad u_2(a, y) = G(y), \quad 0 < y < b$
$u_1(x, 0) = f(x), \quad u_1(x, b) = g(x), \quad 0 < x < a$	$u_2(x, 0) = 0, \quad u_2(x, b) = 0, \quad 0 < x < a$

Suppose u_1 and u_2 are the solutions of Problems 1 and 2, respectively. If we define $u(x, y) = u_1(x, y) + u_2(x, y)$, it is seen that u satisfies all boundary conditions in the original problem (11). For example,

$$\begin{aligned} u(0, y) &= u_1(0, y) + u_2(0, y) = 0 + F(y) = F(y) \\ u(x, b) &= u_1(x, b) + u_2(x, b) = g(x) + 0 = g(x) \end{aligned}$$

and so on. Furthermore, u is a solution of Laplace's equation by Theorem 12.1. In other words, by solving Problems 1 and 2 and adding their solutions, we have solved the original problem. This additive property of solutions is known as the superposition principle. See Figure 12.15.

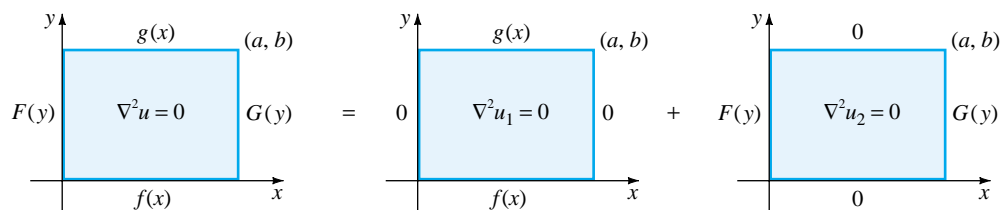


FIGURE 12.15 Solution $u =$ Solution u_1 of Problem 1 + Solution u_2 of Problem 2

We leave to show that a solution of Problem 1 is

$$u_1(x, y) = \sum_{n=1}^{\infty} \left\{ A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y \right\} \sin \frac{n\pi}{a} x,$$

where $A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x dx$

$$B_n = \frac{1}{\sinh \frac{n\pi}{a} b} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x dx - A_n \cosh \frac{n\pi}{a} b \right),$$

and that a solution of Problem 2 is

$$u_2(x, y) = \sum_{n=1}^{\infty} \left\{ A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x \right\} \sin \frac{n\pi}{b} y,$$

where $A_n = \frac{2}{b} \int_0^b F(y) \sin \frac{n\pi}{b} y dy$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b} a} \left(\frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y dy - A_n \cosh \frac{n\pi}{b} a \right).$$