When heat is generated at a rate r within a rod of finite length, the heat equation takes on the form

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0. \tag{1}$$

Equation (1) is nonhomogeneous and is readily shown not to be separable. On the other hand, suppose we wish to solve the homogeneous heat equation $ku_{xx} = u_t$ when the boundary conditions at $x = 0$ and $x = L$ are nonhomogeneous—say, the boundaries are held at nonzero temperatures: $u(0, t) = u_0$ and $u(L, t) = u_1$. Even though the substitution $u(x, t) = X(t)T(t)$ separates $ku_{xx} = u_t$, we quickly find ourselves at an impasse in determining eigenvalues and eigenfunctions since no conclusion can be drawn about $X(0)$ and $X(L)$ from $u(0, t) = X(0)T(t) = u_0$ and $u(L, t) = X(L)T(t) = u_1$.

What follows are two solution methods that are distinguished by different types of nonhomogeneous BVPs.

Consider a BVP involving a *time-independent nonhomogeneous equation* and *time-independent boundary conditions* such as

$$\begin{aligned} &k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \\ &u(0, t) = u_0, \quad u(L, t) = u_1, \quad t > 0 \\ &u(x, 0) = f(x), \quad 0 < x < L, \end{aligned} \tag{2}$$

where u_0 and u_1 are constants. By changing the dependent variable u to a new dependent variable v by the substitution $u(x, t) = v(x, t) + \psi(x)$, the problem in (2) can be reduced to two problems:

$$\text{Problem } A: \quad \{k\psi'' + F(x) = 0, \quad \psi(0) = u_0, \quad \psi(L) = u_1$$

$$\text{Problem } B: \quad \begin{cases} k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}, \\ v(0, t) = 0, \qquad v(L, t) = 0 \\ v(x, 0) = f(x) - \psi(x) \end{cases}$$

Notice that Problem A involves an ODE that can be solved by integration, whereas Problem B is a homogeneous BVP that is solvable by the usual separation of variables. A solution of the original problem (2) is the sum of the solutions of Problems A and B.

The following example illustrates this first method.

Suppose r is a positive constant. Solve equation (1) subject to

$$\begin{aligned} &u(0, t) = 0, \qquad u(1, t) = u_0, \quad t > 0 \\ &u(x, 0) = f(x), \quad 0 < x < 1. \end{aligned}$$

SOLUTION Both the partial differential equation and the boundary condition at $x = 1$ are nonhomogeneous. If we let $u(x, t) = v(x, t) + \psi(x)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi'' \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}.$$

Substituting these results into (1) gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' + r = \frac{\partial v}{\partial t}. \tag{3}$$

Equation (3) reduces to a homogeneous equation if we demand that ψ satisfy

$$k\psi'' + r = 0 \quad \text{or} \quad \psi'' = -\frac{r}{k}.$$

Integrating the last equation twice reveals that

$$\psi(x) = -\frac{r}{2k}x^2 + c_1 x + c_2. \tag{4}$$

Furthermore,
$$u(0, t) = v(0, t) + \psi(0) = 0$$
$$u(1, t) = v(1, t) + \psi(1) = u_0.$$

We have $v(0, t) = 0$ and $v(1, t) = 0$, provided that

$$\psi(0) = 0 \quad \text{and} \quad \psi(1) = u_0.$$

Applying the latter two conditions to (4) gives, in turn, $c_2 = 0$ and $c_1 = r/2k + u_0$. Consequently,

$$\psi(x) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x.$$

Finally, the initial condition $u(x, 0) = v(x, 0) + \psi(x)$ implies that $v(x, 0) = u(x, 0) - \psi(x) = f(x) - \psi(x)$. Thus to determine $v(x, t)$, we solve the new boundary-value problem

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$
$$v(0, t) = 0, \quad v(1, t) = 0, \quad t > 0$$
$$v(x, 0) = f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x, \quad 0 < x < 1$$

by separation of variables. In the usual manner we find

$$v(x, t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where

$$A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x\right] \sin n\pi x \, dx. \tag{5}$$

A solution of the original problem is obtained by adding $\psi(x)$ and $v(x, t)$:

$$u(x, t) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x, \tag{6}$$

where the coefficients A_n are defined in (5).

Observe in (6) that $u(x, t) \longrightarrow \psi(x)$ as $t \longrightarrow \infty$. In the context of solving forms of the heat equation, ψ is called a **steady-state solution.** Since $v(x, t) \longrightarrow 0$ as $t \longrightarrow \infty$, it is called a **transient solution.**

Another type of problem involves a *time-dependent nonhomogeneous equation* and *homogeneous boundary conditions*. Unlike Method 1, where $u(x, t)$ is found by solving two separate problems, it is possible to find the entire solution of a problem such as

$$k\frac{\partial^2 u}{\partial x^2} + F(x, t) = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$$
$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{7}$$
$$u(x, 0) = f(x), \quad 0 < x < L,$$

by making the assumption that time-dependent coefficients $u_n(t)$ and $F_n(t)$ can be found such that both $u(x, t)$ and $F(x, t)$ in (7) can be expanded in the series

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{L}x \quad \text{and} \quad F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi}{L}x, \tag{8}$$

where $\sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$, are the eigenfunctions of $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$ corresponding to the eigenvalues $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$. The latter problem would have been obtained had separation of variables been applied to the associated homogeneous PDE in (7). In (8), observe that the assumed form for $u(x, t)$ already satisfies the boundary conditions in (7). The basic idea here is to substitute the first series in (8) into the nonhomogeneous PDE in (7), collect terms, and equate the resulting series with the actual series expansion found for $F(x, t)$.

The next example illustrates this method.

Solve
$$\frac{\partial^2 u}{\partial x^2} + (1 - x)\sin t = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$
$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0,$$
$$u(x, 0) = 0, \quad 0 < x < 1.$$

SOLUTION With $k = 1$, $L = 1$, the eigenvalues and eigenfunctions of $X'' + \lambda X = 0$, $X(0) = 0$, $X(1) = 0$ are found to be $\lambda_n = \alpha_n^2 = n^2\pi^2$ and $\sin n\pi x$, $n = 1, 2, 3, \ldots$. If we assume that

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x, \tag{9}$$

then the formal partial derivatives of u are

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} u_n(t)(-n^2\pi^2) \sin n\pi x \quad \text{and} \quad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} u_n'(t) \sin n\pi x. \tag{10}$$

Now the assumption that we can write $F(x, t) = (1 - x)\sin t$ as

$$(1 - x)\sin t = \sum_{n=1}^{\infty} F_n(t) \sin n\pi x$$

implies that

$$F_n(t) = \frac{2}{1}\int_0^1 (1 - x)\sin t \sin n\pi x \, dx = 2\sin t \int_0^1 (1 - x)\sin n\pi x \, dx = \frac{2}{n\pi}\sin t.$$

Hence,
$$(1 - x)\sin t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin t \sin n\pi x. \tag{11}$$

Substituting the series in (10) and (11) into $u_t - u_{xx} = (1 - x)\sin t$, we get

$$\sum_{n=1}^{\infty}\left[u_n'(t) + n^2\pi^2 u_n(t)\right]\sin n\pi x = \sum_{n=1}^{\infty} \frac{2\sin t}{n\pi} \sin n\pi x.$$

To determine $u_n(t)$, we now equate the coefficients of $\sin n\pi x$ on each side of the preceding equality:

$$u_n'(t) + n^2\pi^2 u_n(t) = \frac{2\sin t}{n\pi}.$$

This last equation is a linear first-order ODE whose solution is

$$u_n(t) = \frac{2}{n\pi}\left[\frac{n^2\pi^2 \sin t - \cos t}{n^4\pi^4 + 1}\right] + C_n e^{-n^2\pi^2 t},$$

where C_n denotes the arbitrary constant. Therefore the assumed form of $u(x, t)$ in (9) can be written as the sum of two series:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\frac{n^2\pi^2 \sin t - \cos t}{n^4\pi^4 + 1}\right]\sin n\pi x + \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x. \tag{12}$$

Finally, we apply the initial condition $u(x, 0) = 0$ to (12). By rewriting the resulting expression as one series,

$$0 = \sum_{n=1}^{\infty} \left[\frac{-2}{n\pi(n^4\pi^4 + 1)} + C_n \right] \sin n\pi x,$$

we conclude from this identity that the total coefficient of $\sin n\pi x$ must be zero, and so

$$C_n = \frac{2}{n\pi(n^4\pi^4 + 1)}.$$

Hence from (12) we see that a solution of the given problem is

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n^2\pi^2 \sin t - \cos t}{n(n^4\pi^4 + 1)} \sin n\pi x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(n^4\pi^4 + 1)} e^{-n^2\pi^2 t} \sin n\pi x.$$