The temperature in a rod of unit length in which there is heat transfer from its right boundary into a surrounding medium kept at a constant temperature zero is determined from

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -hu(1, t), \quad h > 0, \quad t > 0$$

$$u(x, 0) = 1, \quad 0 < x < 1.$$

Solve for $u(x, t)$.

SOLUTION Proceeding as bef ore with $u(x, t) = X(x)T(t)$ and using $-\lambda$ as the separation constant, we find the separated equations and boundary conditions to be, respectively,

$$X'' + \lambda X = 0 \tag{1}$$

$$T' + k\lambda T = 0 \tag{2}$$

$$X(0) = 0 \quad \text{and} \quad X'(1) = -hX(1). \tag{3}$$

Equation (1) and the homogeneous boundary conditions (3) make up a regular Sturm-Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) + hX(1) = 0. \tag{4}$$

By analyzing the usual three cases in which λ is zero, negative, or positive, we find that only the last case will yield nontrivial solutions. Thus with $\lambda = \alpha^2 > 0$, $\alpha > 0$, the general solution of the DE in (4) is

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x. \tag{5}$$

The first boundary condition in (4) immediately gives $c_1 = 0$. Applying the second condition in (4) to $X(x) = c_2 \sin \alpha x$ yields

$$\alpha \cos \alpha + h \sin \alpha = 0 \quad \text{or} \quad \tan \alpha = -\frac{\alpha}{h}. \tag{6}$$

From the analysis we know that the last equation in (6) has an infinite number of roots. If the consecutive positive roots are denoted α_n, $n = 1, 2, 3, \ldots$, then the eigenvalues of the problem are $\lambda_n = \alpha_n^2$, and the corresponding eigenfunctions are $X(x) = c_2 \sin \alpha_n x$, $n = 1, 2, 3, \ldots$. The solution of the first-order DE (2) is $T(t) = c_3 e^{-k\alpha_n^2 t}$, and so

$$u_n = XT = A_n e^{-k\alpha_n^2 t} \sin \alpha_n x \quad \text{and} \quad u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x.$$

Now at $t = 0$, $u(x, 0) = 1$, $0 < x < 1$, so that

$$1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x. \tag{7}$$

The series in (7) is not a Fourier sine series; rather, it is an expansion of $u(x, 0) = 1$ in terms of the orthogonal functions arising from the regular Sturm-Liouville problem (4). It follows that the set of eigenfunctions $\{\sin \alpha_n x\}$, $n = 1, 2, 3, \ldots$, where the α's are defined by $\tan \alpha = -\alpha/h$, is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[0, 1]$. Matching (7) with (7) of Section 11.1, it follows from (8) of that section, with $f(x) = 1$ and $\phi_n(x) = \sin \alpha_n x$, that the coefficients A_n are given by

$$A_n = \frac{\int_0^1 \sin \alpha_n x \, dx}{\int_0^1 \sin^2 \alpha_n x \, dx}. \tag{8}$$

To evaluate the square norm of each of the eigenfunctions, we use a trigonometric identity:

$$\int_0^1 \sin^2 \alpha_n x\, dx = \frac{1}{2}\int_0^1 (1 - \cos 2\alpha x)\, dx = \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\sin 2\alpha_n\right). \qquad (9)$$

Using the double-angle formula $\sin 2\alpha_n = 2 \sin \alpha_n \cos \alpha_n$ and the first equation in (6) in the form $\alpha_n \cos \alpha_n = -h \sin \alpha_n$, we simplify (9) to

$$\int_0^1 \sin^2\alpha_n x\, dx = \frac{1}{2h}\left(h + \cos^2 \alpha_n\right).$$

Also $$\int_0^1 \sin \alpha_n x\, dx = -\frac{1}{\alpha_n}\cos \alpha_n x\Big|_0^1 = \frac{1}{\alpha_n}(1 - \cos \alpha_n).$$

Consequently, (8) becomes

$$A_n = \frac{2h(1 - \cos \alpha_n)}{\alpha_n(h + \cos^2\alpha_n)}.$$

Finally, a solution of the boundary-value problem is

$$u(x, t) = 2h\sum_{n=1}^{\infty} \frac{1 - \cos \alpha_n}{\alpha_n(h + \cos^2 \alpha_n)} e^{-k\alpha_n^2 t} \sin \alpha_n x.$$

The twist angle $\theta(x, t)$ of a torsionally vibrating shaft of unit length is determined from

$$a^2\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

$$\theta(0, t) = 0, \quad \frac{\partial\theta}{\partial x}\Big|_{x=1} = 0, \quad t > 0$$

$$\theta(x, 0) = x, \quad \frac{\partial\theta}{\partial t}\Big|_{t=0} = 0, \quad 0 < x < 1.$$

See Figure 12.19. The boundary condition at $x = 1$ is called a free-end condition. Solve for $\theta(x, t)$.

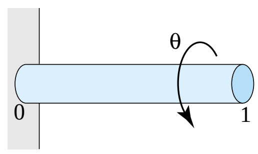

FIGURE 12.19 Twisted shaft

SOLUTION Proceeding as in Section 12.4 with $\theta(x, t) = X(x)T(t)$ and using $-\lambda$ once again as the separation constant, the separated equations and boundary conditions are

$$X'' + \lambda X = 0 \qquad (10)$$

$$T'' + a^2\lambda T = 0 \qquad (11)$$

$$X(0) = 0 \quad \text{and} \quad X'(1) = 0. \qquad (12)$$

A regular Sturm-Liouville problem in this case consists of equation (10) and the homogeneous boundary conditions in (12):

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) = 0. \qquad (13)$$

As in the first example,(13) possesses nontrivial solutions only for $\lambda = \alpha^2 > 0$, $\alpha > 0$. The boundary conditions $X(0) = 0$ and $X'(1) = 0$ applied to the general solution

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad (14)$$

give, in turn, $c_1 = 0$ and $c_2 \cos \alpha = 0$. Since the cosine function is zero at odd multiples of $\pi/2$, $\alpha = (2n - 1)\pi/2$, and the eigenvalues of (13) are $\lambda_n = \alpha_n^2 = (2n - 1)^2\pi^2/4$, $n = 1, 2, 3, \ldots.$ The solution of the second-order DE (11) is $T(t) = c_3 \cos a\alpha_n t + c_4 \sin a\alpha_n t$. The initial condition $T'(0) = 0$ gives $c_4 = 0$, so

$$\theta_n = XT = A_n \cos a\left(\frac{2n - 1}{2}\right)\pi t \sin\left(\frac{2n - 1}{2}\right)\pi x.$$

To satisfy the remaining initial condition, we form

$$\theta(x, t) = \sum_{n=1}^{\infty} A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x. \tag{15}$$

When $t = 0$, we must have, for $0 < x < 1$,

$$\theta(x, 0) = x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)\pi x. \tag{16}$$

As in the first example, the set of eigenfunctions $\left\{\sin\left(\frac{2n-1}{2}\right)\pi x\right\}$, $n = 1, 2, 3, \ldots,$ is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[0, 1]$. Although the series in (16) looks like a Fourier sine series, it is not, because the argument of the sine function is not an integer multiple of $\pi x/L$ (here $L = 1$). The series again is an orthogonal series expansion or generalized Fourier series. Hence from (8) before, the coefficients in (16) are

$$A_n = \frac{\displaystyle\int_0^1 x \sin\left(\frac{2n-1}{2}\right)\pi x\, dx}{\displaystyle\int_0^1 \sin^2\left(\frac{2n-1}{2}\right)\pi x\, dx}.$$

Carrying out the two integrations, we arrive at

$$A_n = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}.$$

The twist angle is then

$$\theta(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x. \tag{17}$$