Suppose the rectangular region in Figure 12.23(a) is a thin plate in which the temperature u is a function of time t and position (x, y). Then, under suitable conditions, $u(x, y, t)$ can be shown to satisfy the **two-dimensional heat equation**

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}. \tag{1}$$

On the other hand, suppose Figure 12.23(b) represents a rectangular frame over which a thin flexible membrane has been stretched (a rectangular drum). If the membrane is set in motion, then its displacement u, measured from the xy-plane (transverse vibrations), is also a function of t and position (x, y). When the vibrations are small, free, and undamped, $u(x, y, t)$ satisfies the **two-dimensional wave equation**

$$a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}. \tag{2}$$

To separate variables in (1) and (2), we assume a product solution of the form $u(x, y, t) = X(x)Y(y)T(t)$. We note that

$$\frac{\partial^2 u}{\partial x^2} = X''YT, \quad \frac{\partial^2 u}{\partial y^2} = XY''T, \quad \text{and} \quad \frac{\partial u}{\partial t} = XYT'.$$

As we see next, with appropriate boundary conditions, boundary-value problems involving (1) and (2) lead to the concept of Fourier series in two variables.

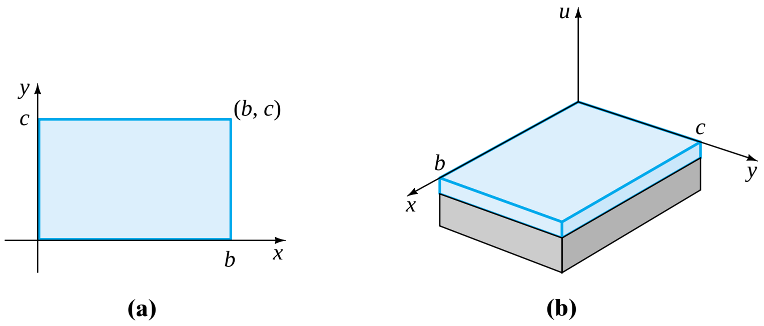

FIGURE 12.23 (a) Rectangular plate and (b) rectangular membrane

Find the temperature $u(x, y, t)$ in the plate shown in Figure 12.23(a) if the initial temperature is $f(x, y)$ throughout and if the boundaries are held at temperature zero for time $t > 0$.

SOLUTION We must solve

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}, \quad 0 < x < b, \quad 0 < y < c, \quad t > 0$$

subject to

$$u(0, y, t) = 0, \quad u(b, y, t) = 0, \quad 0 < y < c, t > 0$$

$$u(x, 0, t) = 0, \quad u(x, c, t) = 0, \quad 0 < x < b, t > 0$$

$$u(x, y, 0) = f(x, y), \quad 0 < x < b, \quad 0 < y < c.$$

Substituting $u(x, y, t) = X(x)Y(y)T(t)$, we get

$$k(X''YT + XY''T) = XYT' \quad \text{or} \quad \frac{X''}{X} = -\frac{Y''}{Y} + \frac{T'}{kT}. \tag{3}$$

Since the left-hand side of the last equation in (3) depends only on x and the right side depends only on y and t, we must have both sides equal to a constant $-\lambda$:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T'}{kT} = -\lambda$$

and so

$$X'' + \lambda X = 0 \tag{4}$$

$$\frac{Y''}{Y} = \frac{T'}{kT} + \lambda. \tag{5}$$

By the same reasoning, if we introduce another separation constant $-\mu$ in (5), then

$$\frac{Y''}{Y} = -\mu \quad \text{and} \quad \frac{T'}{kT} + \lambda = -\mu$$

yield

$$Y'' + \mu Y = 0 \quad \text{and} \quad T' + k(\lambda + \mu)T = 0. \tag{6}$$

Now the homogeneous boundary conditions

$$\left.\begin{array}{ll} u(0, y, t) = 0, & u(b, y, t) = 0 \\ u(x, 0, t) = 0, & u(x, c, t) = 0 \end{array}\right\} \text{ imply that } \left\{\begin{array}{ll} X(0) = 0, & X(b) = 0 \\ Y(0) = 0, & Y(c) = 0. \end{array}\right.$$

Thus we have two Sturm-Liouville problems

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(b) = 0 \tag{7}$$

and

$$Y'' + \mu Y = 0, \quad Y(0) = 0, \quad Y(c) = 0. \tag{8}$$

The usual consideration of cases ($\lambda = 0$, $\lambda = \alpha^2 > 0$, $\lambda = -\alpha^2 < 0$, $\mu = 0$, and so on) leads to two independent sets of eigenvalues

$$\lambda_m = \frac{m^2\pi^2}{b^2} \quad \text{and} \quad \mu_n = \frac{n^2\pi^2}{c^2}.$$

The corresponding eigenfunctions are

$$X(x) = c_2 \sin\frac{m\pi}{b}x, \quad m = 1, 2, 3 \ldots, \quad \text{and} \quad Y(y) = c_4 \sin\frac{n\pi}{c}y, \quad n = 1, 2, 3, \ldots. \tag{9}$$

After substituting the known values of λ_n and μ_n in the first-order DE in (6), its general solution is found to be $T(t) = c_5 e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t}$. A product solution of the two-dimensional heat equation that satisfies the four homogeneous boundary conditions is then

$$u_{mn}(x, y, t) = A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y,$$

where A_{mn} is an arbitrary constant. Because we have two sets of eigenvalues, we are prompted to try the superposition principle in the form of a double sum

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y. \tag{10}$$

At $t = 0$ we must have

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y. \tag{11}$$

We can find the coefficients A_{mn} by multiplying the double sum (11) by the product $\sin(m\pi x/b)\,\sin(n\pi y/c)$ and integrating over the rectangle $0 \le x \le b$, $0 \le y \le c$. It follows that

$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x, y) \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y \, dx\, dy. \tag{12}$$

Thus the solution of the BVP consists of (10) with the A_{mn} defined in (12).

The series (11) with coefficients (12) is called a **sine series in two variables** or a **double sine series.** We summarize next the **cosine series in two variables.**

The **double cosine series** of a function $f(x, y)$ defined over a rectangular region $0 \leq x \leq b$, $0 \leq y \leq c$ is given by

$$\begin{aligned} f(x, y) = A_{00} &+ \sum_{m=1}^{\infty} A_{m0} \cos \frac{m\pi}{b} x + \sum_{n=1}^{\infty} A_{0n} \cos \frac{n\pi}{c} y \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m\pi}{b} x \cos \frac{n\pi}{c} y, \end{aligned}$$

where

$$\begin{aligned} A_{00} &= \frac{1}{bc} \int_0^c \int_0^b f(x, y)\, dx dy \\ A_{m0} &= \frac{2}{bc} \int_0^c \int_0^b f(x, y) \cos \frac{m\pi}{b} x\, dx dy \\ A_{0n} &= \frac{2}{bc} \int_0^c \int_0^b f(x, y) \cos \frac{n\pi}{c} y\, dx dy \\ A_{mn} &= \frac{4}{bc} \int_0^c \int_0^b f(x, y) \cos \frac{m\pi}{b} x \cos \frac{n\pi}{c} y\, dx dy. \end{aligned}$$