

It is likely the origin of the words *even* and *odd* derives from the fact that the graphs of polynomial functions that consist of all even powers of x are symmetric with respect to the y -axis and whereas graphs of polynomials that consist of all odd powers of x are symmetric with respect to origin. For example,

↓ even integer

$$f(x) = x^2 \text{ is even} \quad \text{since } f(-x) = (-x)^2 = x^2 = f(x),$$

↓ odd integer

$$f(x) = x^3 \text{ is odd} \quad \text{since } f(-x) = (-x)^3 = -x^3 = -f(x).$$

See Figures 11.4 and 11.5. The trigonometric cosine and sine functions are even and odd functions, respectively, since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither odd nor even.

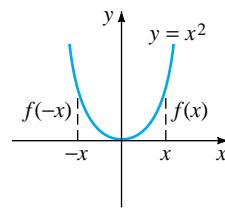


FIGURE 11.4 Even function; graph symmetric with respect to y -axis

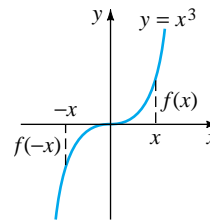


FIGURE 11.5 Odd function; graph symmetric with respect to origin

The following theorem lists some properties of even and odd functions.

THEOREM 11.2 Properties of Even/Odd Functions

- (a) The product of two even functions is even.
- (b) The product of two odd functions is even.
- (c) The product of an even function and an odd function is odd.
- (d) The sum (difference) of two even functions is even.
- (e) The sum (difference) of two odd functions is odd.
- (f) If f is even, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
- (g) If f is odd, then $\int_{-a}^a f(x) dx = 0$.

Let us suppose that f and g are odd functions. Then we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$. If we define the product of f and g as $F(x) = f(x)g(x)$, then

$$F(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = F(x).$$

This shows that the product F of two odd functions is an even function.

If f is an even function on $(-p, p)$, then in view of the foregoing properties the coefficients (9), (10), and (11) of Section 11.2 become

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx = \frac{2}{p} \int_0^p f(x) dx$$

$$a_n = \frac{1}{p} \int_{-p}^p \underbrace{f(x) \cos \frac{n\pi}{p} x}_{\text{even}} dx = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^p \underbrace{f(x) \sin \frac{n\pi}{p} x}_{\text{odd}} dx = 0.$$

Similarly, when f is odd on the interval $(-p, p)$,

$$a_n = 0, \quad n = 0, 1, 2, \dots, \quad b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx.$$

We summarize the results in the following definition.

DEFINITION 11.6 Fourier Cosine and Sine Series

(i) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x, \quad (1)$$

where
$$a_0 = \frac{2}{p} \int_0^p f(x) \, dx \quad (2)$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x \, dx. \quad (3)$$

(ii) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x, \quad (4)$$

where
$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx. \quad (5)$$

Expand $f(x) = x$, $-2 < x < 2$ in a Fourier series.

SOLUTION Inspection of Figure 11.6 shows that the given function is odd on the interval $(-2, 2)$, and so we expand f in a sine series. With the identification $2p = 4$ we have $p = 2$. Thus (5), after integration by parts, is

$$b_n = \int_0^2 x \sin \frac{n\pi}{2} x \, dx = \frac{4(-1)^{n+1}}{n\pi}.$$

Therefore
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} x. \quad (6)$$

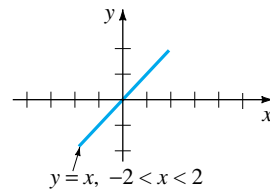


FIGURE 11.6 Odd function in Example 1

The function in the example satisfies the conditions of Theorem 11.1. Hence the series (6) converges to the function on $(-2, 2)$ and the periodic extension (of period 4) given in Figure 11.7.

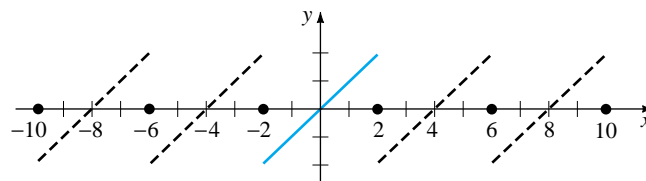


FIGURE 11.7 Periodic extension of function shown in Figure 11.6

With the aid of a CAS we have plotted the graphs $S_1(x)$, $S_2(x)$, $S_3(x)$, and $S_{15}(x)$ of the partial sums of nonzero terms of (7) in Figure 11.9. As seen in Figure 11.9(d), the graph of $S_{15}(x)$ has pronounced spikes near the discontinuities at $x = 0$, $x = \pi$, $x = -\pi$, and so on. This “overshooting” by the partial sums S_N from the functional values near a point of discontinuity does not smooth out but remains fairly constant, even when the value N is taken to be large. This behavior of a Fourier series near a point at which f is discontinuous is known as the **Gibbs phenomenon**.

The periodic extension of f in the example onto the entire x -axis is a meander function.

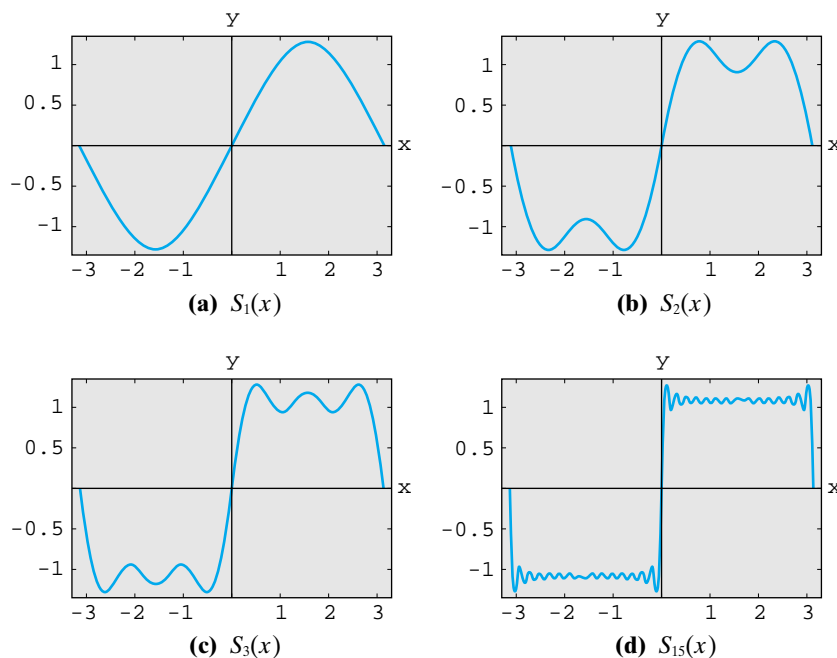


FIGURE 11.9 Partial sums of sine series (7)

Expand $f(x) = x^2$, $0 < x < L$,

(a) in a cosine series (b) in a sine series (c) in a Fourier series.

SOLUTION The graph of the function is given in Figure 11.13.

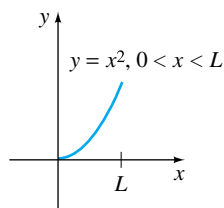


FIGURE 11.13 Function is neither odd nor even.

(a) We have

$$a_0 = \frac{2}{L} \int_0^L x^2 dx = \frac{2}{3} L^2, \quad a_n = \frac{2}{L} \int_0^L x^2 \cos \frac{n\pi}{L} x dx = \frac{4L^2(-1)^n}{n^2\pi^2},$$

where integration by parts was used twice in the evaluation of a_n .

$$\text{Thus} \quad f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{L} x. \quad (8)$$

(b) In this case we must again integrate by parts twice:

$$b_n = \frac{2}{L} \int_0^L x^2 \sin \frac{n\pi}{L} x dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3} [(-1)^n - 1].$$

Hence
$$f(x) = \frac{2L^2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2} [(-1)^n - 1] \right\} \sin \frac{n\pi}{L} x. \quad (9)$$

(c) With $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$, we have

$$a_0 = \frac{2}{L} \int_0^L x^2 dx = \frac{2}{3} L^2, \quad a_n = \frac{2}{L} \int_0^L x^2 \cos \frac{2n\pi}{L} x dx = \frac{L^2}{n^2\pi^2},$$

and
$$b_n = \frac{2}{L} \int_0^L x^2 \sin \frac{2n\pi}{L} x dx = -\frac{L^2}{n\pi}.$$

Therefore
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2\pi} \cos \frac{2n\pi}{L} x - \frac{1}{n} \sin \frac{2n\pi}{L} x \right\}. \quad (10)$$

The series (8), (9), and (10) converge to the $2L$ -periodic even extension of f , the $2L$ -periodic odd extension of f , and the L -periodic extension of f , respectively. The graphs of these periodic extensions are shown in Figure 11.14.

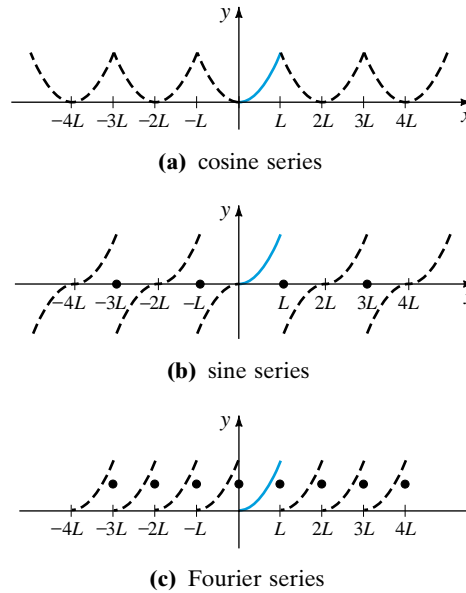


FIGURE 11.14 Same function on $(0, L)$ but different periodic extensions

Fourier series are sometimes useful in determining a particular solution of a differential equation describing a physical system in which the input or driving force $f(t)$ is periodic. In the next example we find a particular solution of the differential equation

$$m \frac{d^2x}{dt^2} + kx = f(t) \quad (11)$$

by first representing f by a half-range sine expansion and then assuming a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{p} t. \quad (12)$$

An undamped spring/mass system, in which the mass $m = \frac{1}{16}$ slug and the spring constant $k = 4$ lb/ft, is driven by the 2-periodic external force $f(t)$ shown in Figure 11.15. Although the force $f(t)$ acts on the system for $t > 0$, note that if we extend the graph of the function in a 2-periodic manner to the negative t -axis,

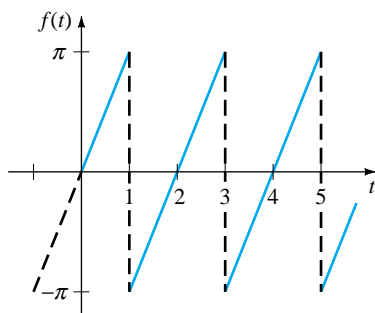


FIGURE 11.15 Periodic forcing function for spring/mass system

we obtain an odd function. In practical terms this means that we need only find the half-range sine expansion of $f(t) = \pi t$, $0 < t < 1$. With $p = 1$ it follows from (5) and integration by parts that

$$b_n = 2 \int_0^1 \pi t \sin n\pi t \, dt = \frac{2(-1)^{n+1}}{n}.$$

From (11) the differential equation of motion is seen to be

$$\frac{1}{16} \frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t. \quad (13)$$

To find a particular solution $x_p(t)$ of (13), we substitute (12) into the equation and equate coefficients of $\sin n\pi t$. This yields

$$\left(-\frac{1}{16} n^2 \pi^2 + 4\right) B_n = \frac{2(-1)^{n+1}}{n} \quad \text{or} \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2 \pi^2)}.$$

Thus

$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2 \pi^2)} \sin n\pi t. \quad (14)$$

Observe in the solution (14) that there is no integer $n \geq 1$ for which the denominator $64 - n^2 \pi^2$ of B_n is zero. In general, if there is a value of n , say N , for which $N\pi/p = \omega$, where $\omega = \sqrt{k/m}$, then the system described by (11) is in a state of pure resonance. In other words, we have pure resonance if the Fourier series expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ (or $\cos(N\pi/L)t$) that has the same frequency as the free vibrations.

Of course, if the $2p$ -periodic extension of the driving force f onto the negative t -axis yields an even function, then we expand f in a cosine series.